Math 3506 2010.

Qu1. (a) Predator-Prey with intraspecific competition.
x = prey, y = predator.

(b) Steady states are $(0,0)$, $(\frac{a}{b}, 0)$ and solutions to

$$a - bx - cy = 0, \quad -d + ex - fy = 0.$$

$$\Rightarrow \begin{pmatrix} e & -f \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d \\ a \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ec+bf}\begin{pmatrix} c & f \\ -b & e \end{pmatrix}\begin{pmatrix} d \\ a \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ec+bf}\begin{pmatrix} cd+af \\ ae-bd \end{pmatrix}$$ which exists only when $\underline{ae > bd}$.

For stability, $J = \begin{pmatrix} a-2bx-cy & -cx \\ ey & -d-2fy+ex \end{pmatrix}$.

Hence $J_{(0,0)} = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix} \Rightarrow (0,0)$ is a saddle.

$$J_{(\frac{a}{b},0)} = \begin{pmatrix} a - \frac{2ba}{b} & -\frac{ca}{b} \\ 0 & \frac{ea}{b} - d \end{pmatrix} = \begin{pmatrix} -a & -\frac{ac}{b} \\ 0 & \frac{1}{b}(ea-bd) \end{pmatrix}$$

$\Rightarrow$ eigenvalues are $-a<0$ and $\frac{1}{b}(ea-bd) < 0$ if $ea < bd$, >0 if $ea > bd$.

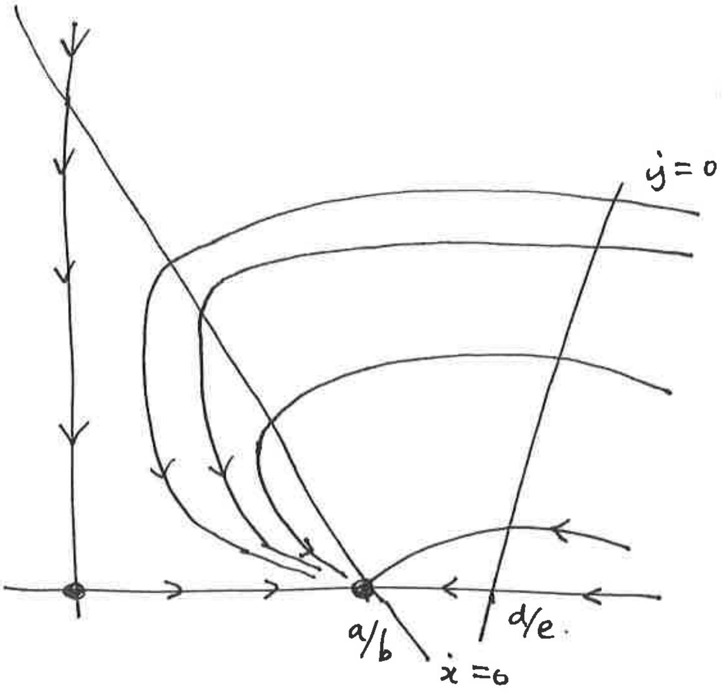
Hence $(\frac{a}{b}, 0)$ is a stable node if $ea < bd$ and a saddle if $ac > bd$.

$$J_{(\bar{x},\bar{y})} = \begin{pmatrix} -b\bar{x} & -c\bar{x} \\ e\bar{y} & -f\bar{y} \end{pmatrix} \Rightarrow \begin{array}{l} \lambda_1 + \lambda_2 = -b\bar{x} - f\bar{y} \\ \lambda_1\lambda_2 = \bar{x}\bar{y}(bf+ce) > 0. \end{array}$$

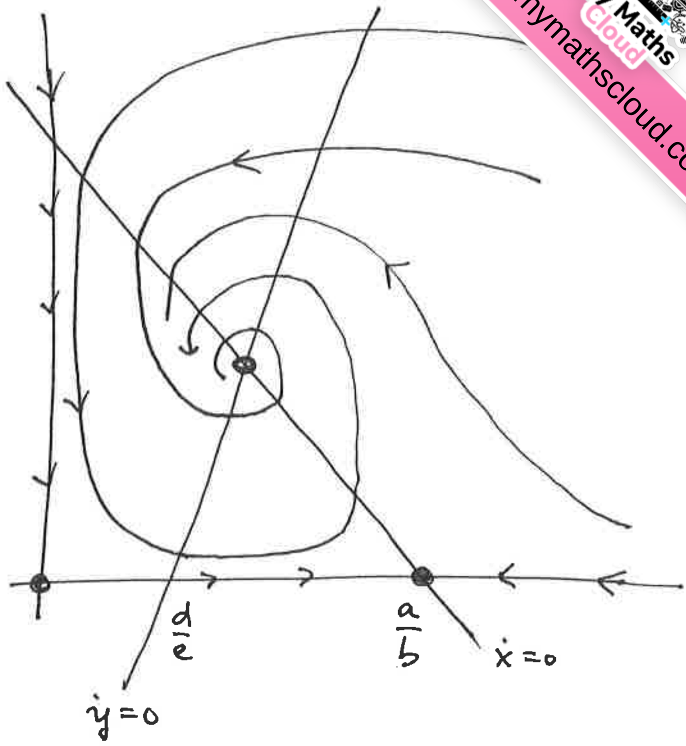
$\Rightarrow (\bar{x},\bar{y})$, when it exists, $(ae > bd)$ is $\underline{\text{locally stable}}$.

Qu 1 contd

(d)



$ae < bd$



$ae > bd.$

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2. (a) The generations are non-overlapping; i.e. the expected remaining lifespan of a sexually mature individual is less than one generation.

(b) Steady states N satisfy $N = rN\exp\left(1-\frac{N}{K}\right)$

i.e. $N=0$ or $\frac{1}{r} = \exp\left(1-\frac{N}{K}\right) \Rightarrow N^* = K(1+\log r)$.

The latter steady state exists only if $r>e$.

(c) Let $f(N) = rN\exp\left(1-\frac{N}{K}\right)$.

$f'(N) = r\exp\left(1-\frac{N}{K}\right) + rN\left(-\frac{1}{K}\right)\exp\left(1-\frac{N}{K}\right)$ $\Rightarrow f'(0) = re > 0 \Rightarrow N=0$ is stable if $re<1$ i.e. $r < 1/e$.

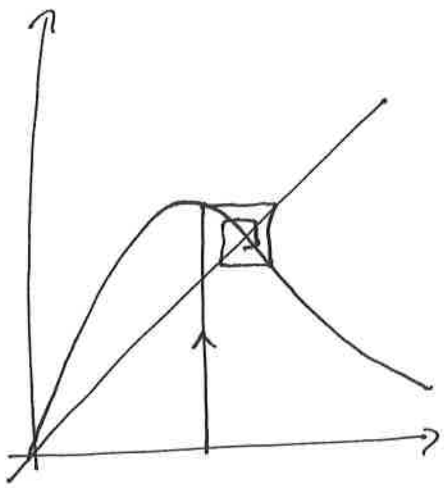
$f'(N^*) = r.\frac{1}{r} + K(\log r + 1)\left(-\frac{1}{K}\right)$

$= -\log r$.

Thus $-1 < -\log r < 1$ provided $r \in \left(\frac{1}{e}, e\right)$ ie stable

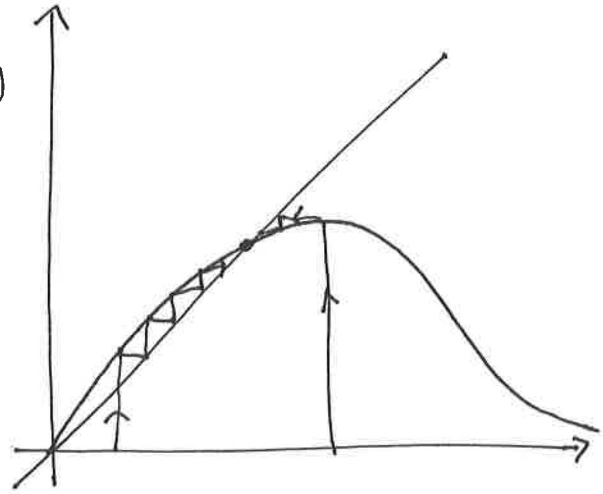
(d)

(i)



$r \in [1, e)$

(ii)



$r \in \left(\frac{1}{e}, 1\right)$.

(e) A stable limit cycle appears.

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Qu 3 (a) ρ = linear ~~birth~~ net reproductive rate

$K(t)$ = carrying capacity of the environment.

(b) $N = Me^{\rho t} \Rightarrow \dot{N} = \dot{M}e^{\rho t} + \rho M e^{\rho t}$

so $(\dot{M} + \rho M)e^{\rho t} = \rho M e^{\rho t}\left(1 - \frac{Me^{\rho t}}{K(t)}\right)$

$$\Rightarrow \dot{M}e^{\rho t} = -\frac{\rho}{K(t)}M^2 e^{2\rho t} \Rightarrow \dot{M} = -\frac{\rho}{K(t)}M^2 e^{\rho t}$$

Hence $$\int_{M_0}^{M} \frac{dM}{M^2} = -\int_0^t \frac{\rho e^{\rho t}}{K(t)}\,dt \Rightarrow \left[\frac{1}{M}\right]_M^{M_0} = -\int_0^t \rho e^{\rho t}/K(t)\,dt$$

Hence $$\frac{1}{M_0} - \frac{1}{M} = -\int_0^t \frac{\rho e^{\rho\tau}}{K(\tau)}\,d\tau$$

Rearranging, $$M = \frac{M_0}{1 + M_0\int_0^t \frac{\rho e^{\rho\tau}}{K(\tau)}\,d\tau}.$$

But $M_0 = N_0$ and $M(t) = e^{-\rho t}N(t)$ giving,

$$N(t) = \frac{N_0 e^{\rho t}}{1 + N_0\int_0^t \frac{\rho e^{\rho\tau}}{K(\tau)}\,d\tau} = \frac{N_0 e^{\rho t}}{1 + N_0\, G(t)}.$$

(c) $$G(kT+s) = \int_0^{kT+s}\frac{\rho e^{\rho\tau}}{K(\tau)}\,d\tau = \int_0^{kT}\frac{\rho e^{\rho\tau}}{K(\tau)}\,d\tau + \int_{kT}^{kT+s}\frac{\rho e^{\rho\tau}}{K(\tau)}\,d\tau$$

$$= \sum_{r=0}^{k-1}\rho\int_{rT}^{(r+1)T}\frac{e^{\rho\tau}}{K(\tau)}\,d\tau + \int_0^s \frac{\rho\, e^{\rho(u+kT)}}{K(u+kT)}\,du$$

Qu3 contd

$$= \sum_{r=0}^{k-1} \rho \int_0^T \frac{e^{\rho(v+rT)}}{K(v+rT)}\, dv + e^{\rho kT} \int_0^s \frac{\rho e^{\rho u}}{K(u)}\, du$$

$$= \sum_{r=0}^{k-1} \rho e^{\rho r T} \int_0^T \frac{e^{\rho v}}{K(v)}\, dv + e^{\rho kT} G(s)$$

$$= \sum_{r=0}^{k-1} e^{\rho r T} . G(T) + e^{\rho kT} G(s)$$

$$= \left(\frac{1 - e^{\rho kT}}{1 - e^{\rho T}} \right) G(T) + e^{\rho kT} G(s) .$$

(d) $$N(kT+s) = \frac{N_0 e^{\rho kT} . e^{\rho s}}{1 + N_0 \left[\left(\frac{1-e^{\rho kT}}{1-e^{\rho T}} \right) G(T) + e^{\rho kT} G(s) \right]}$$

$$= \frac{N_0 e^{\rho s}}{e^{-\rho kT} + N_0 \left[\left(\frac{e^{-\rho kT} - 1}{1 - e^{\rho T}} \right) G(T) + G(s) \right]} .$$

$$\to \frac{e^{\rho s}}{G(s) - \frac{1}{1-e^{\rho T}} G(T)} = N_\infty(s) \text{ as } k \to \infty .$$

Now $N_\infty(0) = (e^{\rho T} - 1)/G(T)$ and $N_\infty(T) = \dfrac{e^{\rho T}}{G(T)\left(1 - \frac{1}{1-e^{\rho T}}\right)}$

$$= (e^{\rho T} - 1)/G(T) = N_\infty(0)$$

$\Rightarrow$ periodic

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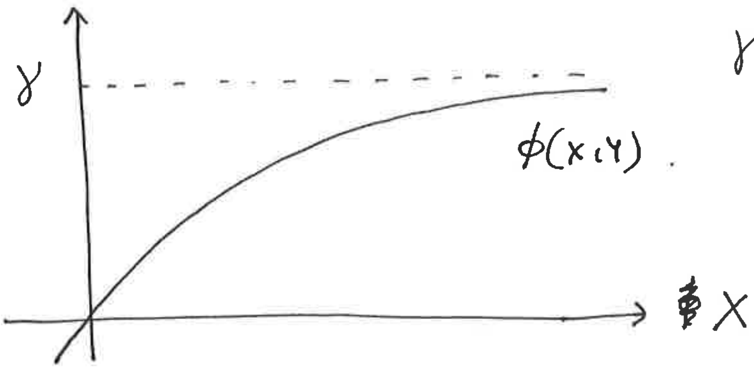
4. (a) X = prey, Y = predator.

(b)



γ = maximum feeding rate.

(c) Steady states: $Y=0$, $X=0$ and $X=K$.

$\Rightarrow$ $(0,0)$ and $(K,0)$ are steady states.

Also when $X = \mu/\sigma$, we have

$$0 = \rho\left(1-\frac{X}{K}\right) - \frac{\gamma Y}{A+X} \Rightarrow Y = \frac{\rho}{\gamma}(A+X)\left(1-\frac{X}{K}\right)$$
$$= \frac{\rho}{\gamma}\left(A+\frac{\mu}{\sigma}\right)\left(1-\frac{\mu}{K\sigma}\right)$$

For this interior steady state we need $\underline{K\sigma > \mu}$.

<u>Stability</u>

$$J = \begin{pmatrix} \rho\left(1-\frac{X}{K}\right) - \frac{\gamma Y}{A+X} - \rho\frac{X}{K} + \frac{\gamma XY}{(A+X)^2} & -\frac{\gamma X}{A+X} \\ \sigma Y & \sigma X - \mu \end{pmatrix}.$$

Hence $J_{(0,0)} = \begin{pmatrix} \rho & 0 \\ 0 & -\mu \end{pmatrix} \Rightarrow (0,0)$ is <u>saddle</u>.

$J_{(K,0)} = \begin{pmatrix} -\rho & -\frac{\gamma K}{A+K} \\ 0 & \sigma K - \mu \end{pmatrix} \Rightarrow$ eigenvalues are $-\rho < 0$

and $\sigma K - \mu < 0$ if $K\sigma < \mu$ (<u>stable node</u>)

and $\sigma K - \mu > 0$ if $K\sigma > \mu$ (<u>saddle</u>)

(Qu 4 contd.)

At the interior steady state (case $K\sigma > \mu$)

$$J = \begin{pmatrix} \dfrac{\gamma X Y}{(A+X)^2} - \dfrac{\rho X}{K} & -\dfrac{\gamma X}{A+X} \\ \sigma Y & 0 \end{pmatrix}$$

so $\lambda_1\lambda_2 = \dfrac{\sigma\gamma X Y}{A+X} > 0$, $\lambda_1+\lambda_2 = X\left[\dfrac{\gamma Y}{(A+X)^2} - \dfrac{\rho}{K}\right]$.

This last expression simplifies using that $\dfrac{\gamma Y}{A+X} = \rho\left(1-\dfrac{X}{K}\right)$

$$\Rightarrow \lambda_1+\lambda_2 = X\left[\frac{\rho\left(1-\frac{X}{K}\right)}{A+X} - \frac{\rho}{K}\right] = \frac{\rho X}{A+X}\left[1-\frac{X}{K}-\frac{A}{K}-\frac{X}{K}\right]$$

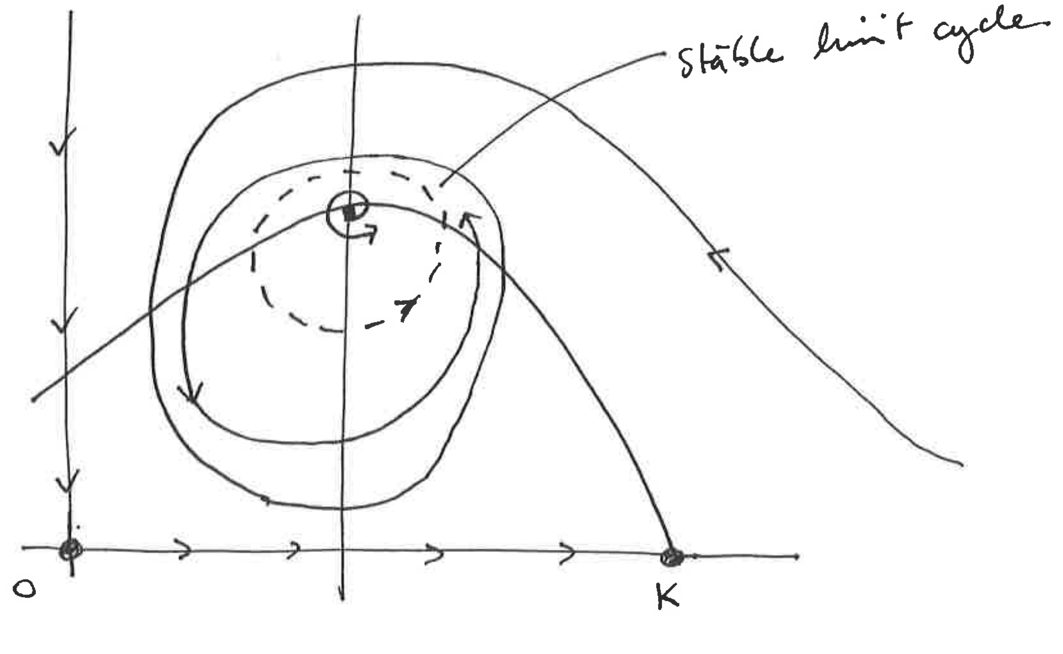
$$= \frac{\rho X}{K(A+X)}\left(K-A-2X\right).$$

But $X = \mu/\sigma$ $\Rightarrow$ $\lambda_1+\lambda_2 = \dfrac{\rho\mu}{K(A\sigma+\mu)}\left(K-A-\dfrac{2\mu}{\sigma}\right)$.

For $A < K - \dfrac{2\mu}{\sigma}$, $\lambda_1+\lambda_2 > 0$ so interior steady state is unstable. For $A > K - 2\mu/\sigma$ interior steady state is locally stable.

(d) We need $K > 2\mu/\sigma$ for changes in A to give rise to a Hopf Bifurcation where A decreases through the critical value $A_c = K - 2\mu/\sigma$.

(e)



5. (a) $N_{k+1}(t+1) = p_k N_k(t) \qquad (k = 0, 1, 2 \ldots)$

$N_0(t) = \sum_{r=1}^{n} b_r N_r(t)$ newborns at t, which survive to age 1 with probability p_0:

$N_1(t+1) = p_0 N_0(t) = \sum_{r=1}^{n} p_0 b_r N_r(t) = \sum_{r=1}^{n} f_r N_r(t).$

where $f_r = p_0 b_r$.

Hence

$$L = \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ p_1 & 0 & & 0 \\ 0 & p_2 & \cdots & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & p_{n-1} & 0 \end{pmatrix}$$

(ii) Eigenvalues λ satisfy $|L - \lambda I| = 0$. Let

$$\phi_n = \begin{vmatrix} f_1-\lambda & f_2 & \cdots & f_n \\ p_1 & -\lambda & \cdots & 0 \\ 0 & p_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & p_{n-1} & -\lambda \end{vmatrix} = (-\lambda) \begin{vmatrix} f_1-\lambda & f_2 & \cdots & f_{n-1} \\ p_1 & -\lambda & \cdots & 0 \\ & \ddots & & \\ 0 & \cdots & p_{n-2} & -\lambda \end{vmatrix}$$

$$+ (-1)^{n-1} f_n (p_1 \cdots p_{n-1}).$$

$$= -\lambda \phi_{n-1} + (-1)^{n-1} b_n \prod_{r=0}^{n-1} p_r . = -\lambda \phi_{n-1} + (-1)^{n-1} b_n l_n$$

where $l_s = \prod_{r=0}^{s-1} p_r$ is the probability of surviving from birth to age s.

$$\phi_n = (-\lambda)\phi_{n-1} + (-1)^{n-1} b_n l_n$$
$$= (-\lambda)\left[-\lambda \phi_{n-2} + (-1)^{n-2} b_{n-1} l_{n-1}\right] = (-\lambda)^2 \phi_{n-2} + (-1)^{n-1}(\lambda b_{n-1} l_{n-1} + b_n l_n)$$
$$+ (-1)^{n-1} b_n l_n$$

(Qu 5 contd).

$$= -\lambda^2\left[(-\lambda)\phi_{n-3} + (-1)^{n-3} b_{n-2} l_{n-2}\right] + (-1)^{n-1}\left(\lambda b_{n-1} l_{n-1} + b_n l_n\right)$$

$$= (-\lambda)^3 \phi_{n-3} + (-1)^{n-1}\left(\lambda^2 b_{n-2} l_{n-2} + \lambda b_{n-1} l_{n-1} + b_n l_n\right)$$

... and so by induction,

$$\phi_n = (-\lambda)^{n-1} . \phi_1 + (-1)^{n-1} \sum_{r=0}^{n-2} \lambda^r b_{n-r} l_{n-r} .$$

But $\phi_1 = f_1 - \lambda$ so eigenvalues satisfy $0 = \phi_n$, using $f_1 = p_0 b_1$

$$\Rightarrow \cancel{(-1)^{n-1}} \lambda^{n-1}(p_0 b_1 - \lambda) + \cancel{(-1)^{n-1}} \sum_{r=0}^{n-2} b_{n-r} l_{n-r} \lambda^r = 0 .$$

$$-\lambda^n + \lambda^{n-1} l_1 b_1 + l_2 b_2 \lambda^{n-2} + \cdots + \lambda b_{n-1} l_{n-1} + b_n l_n = 0$$

which gives for non-zero eigenvalues,

$$\sum_{r=1}^{n} \frac{b_r l_r}{\lambda^r} = 1 .$$

(c) Let $G(\lambda) = \sum_{r=1}^{n} b_r l_r / \lambda^r$. Then on $\lambda > 0$ the function is monotonically decreasing from $+\infty$ to 0. Hence $G(\lambda) = 1$ has a unique root $\lambda_0 > 0$.

(d) $\underline{N}_0 = \sum \alpha_i(0) \underline{v}_i$ say, $\underline{v}_i$ eigenvector for eigenvalue λ_i.

$\underline{N}_{t+1} = L \underline{N}_t \Rightarrow \underline{N}_t = L^t\left(\sum \alpha_i(0) \underline{v}_i\right) = \sum \alpha_i(0) \lambda_i^t \underline{v}_i$

$= \lambda_0^t \sum \alpha_i \left(\frac{\lambda_i}{\lambda_0}\right)^t \underline{v}_i \longrightarrow \lambda_0^t \alpha_0(0) \underline{v}_0$ for large t, since $\lambda_0 > |\lambda_i|$ by aperiodicity of $\underline{L}$. Hence $\underline{N}(t+1) \simeq \lambda_0 \underline{N}(t)$ for large t.

(Qn 5 contd)

(e). If $\sum b_r l_r = 1$ then $\lambda = 1$ solves $G(\lambda) = 1$.

By periodicity of L, $\lambda = 1$ is the (unique) dominant eigenvalue of L and $\lambda_0 = 1 \Rightarrow \underline{N}(t+1) \simeq \underline{N}(t)$ for large t, i.e. the population $\underline{N}(t)$ tends to an equilibrium.